Systematic perturbation theory for sine - Gordon solitons without use of inverse scattering methods

1997 J. Phys. A: Math. Gen. 30 1227

(http://iopscience.iop.org/0305-4470/30/4/023)

J. Phys. A: Math. Gen. **30** (1997) 1227–1241. Printed in the UK PII: S0305-4470(97)74766-8

Systematic perturbation theory for sine–Gordon solitons without use of inverse scattering methods

E Mann

Max-Planck-Institut für Metallforschung, Institut für Physik, PO Box 800665, D-70506 Stuttgart, Germany

Received 20 May 1996, in final form 23 September 1996

Abstract. The perturbed sine–Gordon soliton solutions (kinks, breathers) in laboratory coordinates are derived by a Green function formalism in a novel way dispensing with any inverse scattering methods. Besides the Bäcklund transformation, a simple approach based on Green's theorem is employed for determining the Green functions. A systematic treatment allows the adiabatic approximation to be derived in a new manner. The results are compared to those of the relevant papers in this field and thereby further insight is provided. In a simple example (a kink scattered by an impurity) the calculation of energy radiation is demonstrated.

1. Introduction

The nonlinear partial differential equation named the sine–Gordon equation is one of the most familiar soliton equations. Its simplest solutions, the one-soliton (kink) and the two-soliton (breather) solutions, play an important role in many fields of physics. In applications, the influence of various perturbations on the soliton behaviour is of particular interest.

The history of the activities in the field of perturbed soliton equations is more than 40 years old. In 1951 Seeger and Kochendörfer [1] investigated the influence of weak perturbations on the kink solution of the sine–Gordon equation. Their approach has recently been reformulated in terms of the Bäcklund transformation [2]. Later powerful methods were developed that are based, to a varying extent, on inverse scattering theory [3]. In particular, the pioneering works by McLaughlin and Scott [4], Kaup and Newell [5], and Karpman *et al* [6, 7] are to be noted. Although a strong mathematical tool, inverse scattering theory is rather involved and not easily accessible. Therefore, also other more familiar and simple methods have been looked for. There are some papers dealing with perturbed single solitons, for example [8–10], that do not use inverse scattering methods, but here special assumptions have been made and no general solutions could be given.

It has been our objective, in continuing the early work of Seeger *et al* [1, 11] (cf also [12]) to find general methods that do not need the knowledge and the use of inverse scattering theory; rather we searched for methods that are based on the use of the Bäcklund transformation and are suited to the treatment of perturbed single-soliton as well as multi-soliton solutions of the sine–Gordon equation. Such methods have been given for the perturbed kink solutions [2] and also for the perturbed breather solutions [13, 14]. These methods have in common that first the fundamental solutions of the homogeneous, in the neighbourhood of the unperturbed soliton solution linearized, sine–Gordon equation have to be found, which is achieved by means of the Bäcklund transformation. The main problem

0305-4470/97/041227+15\$19.50

is the determination of the coefficients of an expansion of the perturbed solution in terms of these basis functions. To this end an appropriate equation, called the key equation, has to be found, which is accomplished very easily in the one-soliton case by a simple combination of Bäcklund transformations. In the two-soliton case, however, the key equation becomes much more complicated and may only be obtained by a rather intricate combination of several Bäcklund transformations.

In the present paper a new method is presented that rests on the use of Green's theorem. It allows the above-mentioned coefficients to be determined in a direct way and is, for the two-soliton case, not much more complicated than that for the one-soliton case. Moreover, it is equally well applicable to other perturbed soliton equations, for example the perturbed Korteweg–deVries equation [15]. It seems to us that this method is able to compete with the existing methods depending on the use of inverse scattering theory (compare also with [16–20]) but has the advantage of being more direct and simple.

In the following the general problem is outlined and the formal first-order solution is given in terms of a Green function. In particular, arbitrary initial conditions are allowed for (section 2). In section 3 the basis functions for an expansion of the perturbed solution are determined. This requires the consideration of the Bäcklund transformations for both kinks and breathers. The expansion coefficients are specified from conditions on the Green function by means of Green's theorem (section 4). In section 5 the pure first-order result is improved in the sense of the adiabatic approximation, whereby possible secular terms are eliminated. In an application to a simple example it is shown how the perturbed solution can be utilized to calculate the radiated energy (section 6). The results are compared with corresponding expressions in the literature and some new conclusions are drawn (section 7).

2. General formulation

The normalized sine–Gordon equation with a perturbation term $\varepsilon F(x,t)$, $|\varepsilon| \ll 1$, is defined as

$$u_{tt} - u_{xx} + \sin u = \varepsilon F. \tag{2.1}$$

An unperturbed soliton solution, for example, a kink or a breather, is denoted by $u_s(x,t)$. In a first-order approximation we assume that the perturbed solution may be written as

$$u = u_s + \varepsilon v \tag{2.2}$$

where $\varepsilon v(x,t)$ denotes the deviation from the unperturbed soliton solution. Inserting (2.2) into (2.1) and retaining only the terms linear in ε gives

$$Lv \equiv v_{tt} - v_{xx} + (\cos u_s)v = F \tag{2.3}$$

where we have introduced the linear operator L.

The inhomogeneous linear partial differential equation (2.3) is formally solved by a Green function $G(x,t;x',t')$ defined by

$$LG \equiv G_{tt} - G_{xx} + (\cos u_s)G = \delta(x-x')\delta(t-t'). \tag{2.4}$$

In terms of G a particular solution v of equation (2.3) may be expressed as

$$v = \int_0^{\infty} \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x' G(x,t;x',t')F(x',t'). \tag{2.5}$$

By forming Lv this solution is verified at once.

Because of the causality condition [21], $G(x,t;x',t')$ should be zero for $t<t'$, which is achieved by the ansatz

$$G(x,t;x',t') = G^0(x,t;x',t')H(t-t') \tag{2.6}$$

where $H(\tau)$ designates the step function: $H(\tau)=1$ for $\tau>0$, $H(\tau)=0$ for $\tau<0$. The solution (2.5) then becomes

$$v = \int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x' G^0(x,t;x',t')F(x',t') \equiv v_1(x,t). \tag{2.7}$$

The function G^0 is chosen to be a solution of the homogeneous equation

$$LG^0 = 0. \tag{2.8a}$$

In order that the solution v given in (2.7) fulfills equation (2.3), the Green function G^0 has to satisfy the following conditions:

$$G^0(x,t;x',t')|_{t=t'} = 0 \qquad G^0_t(x,t;x',t')|_{t=t'} = \delta(x-x'). \tag{2.8b}$$

Equations (2.8a) and (2.8b) describe an initial-value problem for the Green function G^0. Once G^0 is determined, the solution v follows by quadratures.

The particular solution $v=v_1$ of (2.7) satisfies the initial conditions $v_1 = v_{1t} = 0$ for $t=0$. If we are looking for a solution satisfying the initial conditions $v(x,0)=f(x)$, $v_t(x,0)=g(x)$, we have to add a solution v_0 of the homogeneous equation

$$v(x,t) = v_0(x,t) + v_1(x,t) \tag{2.9}$$

where

$$Lv_0 = 0 \qquad v_0(x,0) = f(x) \qquad v_{0t}(x,0) = g(x) \tag{2.10}$$

defines the initial-value problem for v_0. In simple cases v_0 may be found directly from (2.10). On the other hand, v_0 may be expressed in terms of the Green function G^0. This is done in section 4, equation (4.11), after the general properties of the Green function G^0 have been established.

3. Solution of the homogeneous equation

In order to solve the initial-value problem (2.8) for the Green function G^0, we first have to find the general solution of the homogeneous equation

$$L\varphi \equiv \varphi_{tt} - \varphi_{xx} + (\cos u_s)\varphi = 0. \tag{3.1}$$

This equation results from the unperturbed sine–Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{3.2}$$

by a variation of u in the neighbourhood of the soliton solution u_s. So one may either solve (3.1) directly or find the general solution of (3.2) in the neighbourhood of u_s and then differentiate with respect to the parameters p_i (which may be discrete or continuous)

$$\varphi_i = (\partial u/\partial p_i)|_{u=u_s}. \tag{3.3}$$

The first procedure is possible for the one-soliton (kink) solution only; for the second procedure, which applies to all soliton solutions, we have to employ the Bäcklund transformation.

3.1. One-soliton (kink) case

The (positive) kink solution of (3.2) is written as

$$u_s = u_k = 4 \arctan \exp \bar{x} \qquad \bar{x} = \frac{x - \sin\sigma \cdot t - x_0}{\cos\sigma} \tag{3.4}$$

where σ and x_0 are two parameters. In the literature the velocity parameter V is frequently introduced: $\sin\sigma = V, \cos\sigma = (1 - V^2)^{1/2}$. The function $\cos u_s$ in the homogeneous equation (3.1) becomes

$$\cos u_k = 1 - 2\,\mathrm{sech}^2\,\bar{x}. \tag{3.5}$$

Since the sine–Gordon equation is Lorentz invariant, the Lorentz transformation

$$\bar{x} = \frac{x - x_0 - \sin\sigma \cdot t}{\cos\sigma} \qquad \bar{t} = \frac{t - \sin\sigma(x - x_0)}{\cos\sigma} \tag{3.6}$$

in (3.1) leads to

$$\varphi_{\bar{t}\bar{t}} - \varphi_{\bar{x}\bar{x}} + (1 - 2\,\mathrm{sech}^2\,\bar{x})\varphi = 0. \tag{3.7}$$

This equation admits a continuous solution with the parameter $\bar{k}$ $(1 + \bar{k}^2 = \bar{\omega}^2)$ and two discrete solutions which, according to (3.3), are also obtained by differentiating the function u_k, (3.4), with respect to the two parameters σ and x_0 [2].

The general solution of (3.1) for the kink soliton then becomes

$$\varphi = A_1\,\mathrm{sech}\,\bar{x} + A_2\bar{t}\,\mathrm{sech}\,\bar{x} + \int_{-\infty}^{+\infty} \mathrm{d}\bar{k}(\tanh\bar{x} - \mathrm{i}\bar{k})[\bar{A}(\bar{k})\mathrm{e}^{\mathrm{i}(\bar{k}\bar{x}+\bar{\omega}\bar{t})} + \bar{B}(\bar{k})\mathrm{e}^{\mathrm{i}(\bar{k}\bar{x}-\bar{\omega}\bar{t})}] \tag{3.8}$$

or, if we return in the exponents to x, t-coordinates by defining $(\omega = +\sqrt{1+k^2})$

$$\bar{k}\bar{x} + \bar{\omega}\bar{t} = k(x - x_0) + \omega t: \qquad \bar{k} = \frac{k + \omega\sin\sigma}{\cos\sigma} \qquad \bar{\omega} = \frac{\omega + k\sin\sigma}{\cos\sigma} \tag{3.9}$$

$$\varphi = (A_1 + A_2\bar{t})\,\mathrm{sech}\,\bar{x} + \int_{-\infty}^{+\infty} \mathrm{d}k\left[A(k)\left\{\frac{\tanh\bar{x} - \mathrm{i}\bar{k}}{\mathrm{i}\bar{\omega}}\mathrm{e}^{\mathrm{i}(kx+\omega t)}\right\} + B(k)\{-\omega\}\right] \tag{3.10}$$

where the symbol $\{-\omega\}$ denotes that in the foregoing expression ω is to be replaced by $-\omega$, and where the constants A and B have been introduced in the way as done for reasons to be seen later.

3.2. Two-soliton (breather) case. Bäcklund transformations

The so-called breather solution of (3.2) may be written as

$$u_s = u_b = -4 \arctan\left\{\frac{1}{\sinh\sigma}\frac{\sin(\tanh\sigma \cdot t + c_2)}{\cosh(\mathrm{sech}\,\sigma \cdot x + c_1)}\right\} \tag{3.11}$$

the minus sign has been chosen to be in accordance with the subsequent Bäcklund transformations. There are three parameters σ, c_1 and c_2. In the literature sometimes $\tanh\sigma = \cos\mu$, $\mathrm{sech}\,\sigma = \sin\mu$, $\mathrm{csch}\,\sigma = \tan\mu$ is written. For brevity, we shall occasionally write $\tanh\sigma = \beta$, $\mathrm{sech}\,\sigma = \alpha$, $\mathrm{csch}\,\sigma = \gamma$; $\gamma = \alpha\beta^{-1}, \alpha^2 + \beta^2 = 1$. The function (3.11) represents a breather whose centre is at rest. A running breather is formally obtained by a Lorentz transformation

$$x = (\tilde{x} - c\tilde{t})(1 - c^2)^{-1/2} \qquad t = (\tilde{t} - c\tilde{x})(1 - c^2)^{-1/2}. \tag{3.12}$$

The constant c is the fourth parameter that defines the most general breather.

We wish to solve the homogeneous equation (3.1) with

$$\cos u_b = 1 - 8\gamma^{-2} N^{-2} \cosh^2 \alpha x \sin^2 \beta t \qquad N = \gamma^{-2} \cosh^2 \alpha x + \sin^2 \beta t \tag{3.13}$$

where we have specified the breather (3.11) by assuming $c_1 = c_2 = 0$. For this task we utilize the Bäcklund transformation. Since there have been many presentations, for example, [4, 11–13, 22, 23], we confine ourselves to the relevant expressions. In terms of characteristic coordinates $p = (x - t)/2$ and $q = (x + t)/2$, the Bäcklund transformation

$$\partial_p (u_i - u_0)/2 = \lambda_i \sin[(u_i + u_0)/2] \qquad \partial_q (u_i + u_0)/2 = \lambda_i^{-1} \sin[(u_i - u_0)/2] \tag{3.14}$$

describes the production of a new solution u_i from a given solution u_0 of the sine–Gordon equation (3.2). For $u_0 = 0$ and with $\lambda_i = (1 + \sin\sigma_i)/\cos\sigma_i$, kink solutions corresponding to (3.4) result, with $\bar{x}_i = (x - \sin\sigma_i \cdot t - x_i)/\cos\sigma_i$, where the x_i denote integration constants. For $u_0 \neq 0$, the solutions u_i represent kinks superimposed on u_0. Starting with the solution u_1, a second Bäcklund transformation with the parameter λ_2 produces a second kink on u_0. By virtue of Bianchi's theorem, this kink-pair solution u_{12} is given in pure algebraic form

$$u_{12} = u_0 + 4 \arctan \left\{ \frac{\cos \frac{1}{2}(\sigma_1 + \sigma_2)}{\sin \frac{1}{2}(\sigma_1 - \sigma_2)} \tan \frac{u_1 - u_2}{4} \right\}. \tag{3.15}$$

For $u_0 = 0, \sigma_1 = -\sigma_2 = \mathrm{i}\sigma, x_1 = x_2 = 0$ the breather solution (3.11) with $c_1 = c_2 = 0$ results. Since for performing the differentiations in (3.3) it suffices to assume $|u_0| \ll 1$, the corresponding solution of (3.2) is given by $u_0 = A(k) \exp[\mathrm{i}(kx + \omega t)] + B(k) \exp[\mathrm{i}(kx - \omega t)]$, where $\omega = +\sqrt{1 + k^2}$. With this form for u_0 the solutions u_i of (3.14) are exactly those given in (3.10) [2].

We are now in a position to derive all solutions of the homogeneous equation (3.1) with (3.13) by means of (3.3) with $u = u_{12}$. The discrete solutions $\varphi_1, \varphi_2, \varphi_3$ and φ_4 are obtained in a simpler way by differentiating the form (3.11) plus (3.12) with respect to c_1, c_2, c and σ, respectively, and then setting $c_1 = c_2 = c = 0$. The continuous solutions follow from (3.15) setting $\sigma_1 = -\sigma_2 = \mathrm{i}\sigma$, $x_1 = x_2 = 0$ in advance and $A = B = 0$ after differentiating with respect to A and B. After all, the most general solution of (3.1) in the breather case may be written, with the abbreviations $\xi = \alpha x$, $\eta = \beta t$, $N = \gamma^{-2} \cosh^2 \xi + \sin^2 \eta$,

$$\varphi(x, t) = \sum_{\mu=1}^{4} A_\mu \varphi_\mu(x, t) + \int_{-\infty}^{+\infty} \mathrm{d}k [A(k)\phi(x, t; k, \omega) + B(k)\phi(x, t; k, -\omega)] \tag{3.16}$$

where

$$\varphi_1 = \frac{1}{N} \sinh\xi \sin\eta \qquad \varphi_3 = \frac{1}{N} [\eta \sinh\xi \sin\eta - \gamma^{-2} \xi \cosh\xi \cos\eta]$$

$$\varphi_2 = \frac{1}{N} \cosh\xi \cos\eta \qquad \varphi_4 = \frac{1}{N} [\alpha^{-2} \cosh\xi \sin\eta - \eta \cosh\xi \cos\eta - \gamma^{-2} \xi \sinh\xi \sin\eta]$$

$$\phi(x, t; k, \omega) = \mathrm{e}^{\mathrm{i}(kx + \omega t)} \left[1 + \frac{2(\sin^2\eta - \cosh^2\xi) + \mathrm{i}(\alpha^{-1} k \sinh 2\xi + \beta^{-1} \omega \sin 2\eta)}{(1 + \alpha^{-2} k^2)\gamma^2 N} \right].$$

4. Determination of the Green function G^0

The Green function G^0 defined through the initial-value problem (2.8) may be expanded in the complete basis derived in the last section:

$$G^0(x, t; x', t') = \sum_i A_i(x', t') \varphi_i(x, t). \tag{4.1}$$

The index i runs over both discrete and continuous basis functions. Since orthogonality relations between the functions φ_i are, in general, unknown (if one does not wish to borrow from inverse scattering theory), we have to look for other methods in order to solve the initial-value problem (2.8). One method has been reported in [2, 13, 14]. It consists of finding special combinations of φ, as given by (3.10) or (3.16), and its derivatives in such a way that the discrete parts disappear and the continuous parts appear as Fourier integrals. This could be achieved by combining several Bäcklund transformations and was much more complicated in the breather case than in the kink case. Here we shall present a new method that is more general, simpler, and applicable to kinks and breathers in about the same way.

Led by Green's theorem, which for the self-adjoint operator L from (2.3) with any two functions $u(x,t)$ and $v(x,t)$ reads

$$uLv - vLu = \frac{\partial}{\partial t}(uv_t - u_t v) - \frac{\partial}{\partial x}(uv_x - u_x v) \tag{4.2}$$

we write the two conditions (2.8b) on G^0 for $t = t'$ as

$$0 = \sum_i A_i \varphi_i(x,t') \qquad \delta(x-x') = \sum_i A_i \varphi_{it'}(x,t') \tag{4.3}$$

multiply the second relation by $\varphi_j^*(x,t')$, the first by $\varphi_{jt'}^*(x,t')$, subtract, integrate over x, and obtain

$$\varphi_j^*(x',t') = \sum_i A_i M_{ij}(t') \qquad M_{ij}(t) = \int_{-\infty}^{+\infty} [\varphi_{it}\varphi_j^* - \varphi_i \varphi_{jt}^*]\,\mathrm{d}x. \tag{4.4}$$

Employing Green's theorem (4.2), with $L\varphi_i = 0$, the elements M_{ij} satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{ij}(t) = [\varphi_{ix}\varphi_j^* - \varphi_i\varphi_{jx}^*]_{x=+\infty} - [\varphi_{ix}\varphi_j^* - \varphi_i\varphi_{jx}^*]_{x=-\infty}. \tag{4.5}$$

The right-hand side of (4.5) is easily evaluated. If i or j denotes a discrete basis function, the right-hand side vanishes, and $M_{ij}(t)$ is zero or a constant. The constant elements M_{ij} are simply calculated from the integral (4.4) by choosing t conveniently, for example, $t = 0$. The only non-trivial elements arise for both i and j denoting continuous basis functions. For example, if we write in the kink case, in correspondence with (3.10), $G^0 = A_1\varphi_1 + A_2\varphi_2 + \int \mathrm{d}k[A(k)\varphi_k^+ + B(k)\varphi_k^-]$, then the element formed with the functions φ_k^+ and $\varphi_{k'}^{+*}$ takes, after a simple t integration, the form (when we omit terms which would vanish with the subsequent k integration)

$$M_{kk'}^{++} \Longrightarrow \lim_{R\to\infty} h(k,k') \frac{\sin(k-k')R}{k-k'} = h(k,k)\pi\delta(k-k') \tag{4.6}$$

where $h(k,k') = 2\mathrm{i}(\omega+\omega')(1+\bar{k}\bar{k}')(\bar{\omega}\bar{\omega}')^{-1}\exp[\mathrm{i}(\omega-\omega')t]$. In the breather case the procedure is quite similar. Because of the δ functions the system (4.4) becomes a finite algebraic system and the coefficients A_i are expressed in the functions $\varphi_j^*(x',t')$ in a simple way. The results are, in detail, as follows.

(1) *Kink case.* $M_{21} = -M_{12} = 2$, $M_{kk'}^{++} = -M_{kk'}^{--} = 4\pi\mathrm{i}\omega\delta(k-k')$. All other elements are zero. Solving the system (4.4), the Green function becomes

$$G^0(x,t;x',t') = \frac{\bar{t}-\bar{t}'}{2\cosh\bar{x}\cosh\bar{x}'} + \int_{-\infty}^{+\infty} \mathrm{d}k \left[\left\{ \frac{(\tanh\bar{x} - \mathrm{i}\bar{k})(\tanh\bar{x}' + \mathrm{i}\bar{k})}{4\pi\mathrm{i}\omega\bar{\omega}^2}\, \mathrm{e}^{\mathrm{i}k(x-x')+\mathrm{i}\omega(t-t')} \right\} + \{-\omega\} \right]. \tag{4.7}$$

(2) *Breather case.* $M_{31} = -M_{13} = M_{42} = -M_{24} = \gamma = \operatorname{csch}\sigma$; $M^{++}_{kk'} = -M^{--}_{kk'} = 4\pi\mathrm{i}\omega\delta(k-k')$. All other elements are zero. With (4.4) and the notation (3.16), the Green function becomes

$$G^0(x,t;x',t') = -\sinh\sigma[\varphi_1\varphi_3' - \varphi_3\varphi_1' + \varphi_2\varphi_4' - \varphi_4\varphi_2'] + \frac{1}{4\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{\omega}[\phi(x,t;k,\omega)\phi^*(x',t';k,\omega) - \phi(x,t;k,-\omega)\phi^*(x',t';k,-\omega)]. \tag{4.8}$$

The above formula applies to the breather whose centre is at rest. The Green function for the running breather with coordinates $\tilde{x}$ and $\tilde{t}$ is obtained by applying the Lorentz transformation (3.12) to both the unprimed and primed coordinates. In doing so it is implied that the elements M_{ij} remain the same. Indeed, it can generally be shown that $\partial M_{ij}/\partial c = 0$ (see the appendix), so that the elements M_{ij} are the same as in the case $c = 0$. As to the continuous part of G^0, it may be appropriate also to transform the variables k, ω: $k = r(\tilde{k} + c\tilde{\omega})$, $\omega = r(\tilde{\omega} + c\tilde{k})$, with $r = (1-c^2)^{-1/2}$. Then, for example, $kx + \omega t$ becomes $\tilde{k}\tilde{x} + \tilde{\omega}\tilde{t}$ and $\mathrm{d}k/\omega$ is to be replaced by $\mathrm{d}\tilde{k}/\tilde{\omega}$.

With the determination of G^0 the problem of finding the solution $v = v_1$, (2.7), for vanishing initial values is formally solved. In the case of non-zero initial values the additional term v_0 defined in (2.10) may be expressed in the basis functions φ_i, and the initial-value problem (2.10) becomes

$$v_0(x,t) = \sum_i C_i\varphi_i(x,t) \qquad f(x) = \sum_i C_i\varphi_i(x,0) \qquad g(x) = \sum_i C_i\varphi_{it}(x,0). \tag{4.9}$$

A similar procedure as from (4.3) to (4.4) leads to

$$\int_{-\infty}^{+\infty}\mathrm{d}x'[g(x')\varphi_j^*(x',0) - f(x')\varphi_{jt'}^*(x',0)] = \sum_i C_i M_{ij}(0). \tag{4.10}$$

Since, as shown above, the relevant elements M_{ij} are time-independent, we may manipulate the system (4.4) in a way that the same left-hand side arises as in (4.10). By comparison, it follows that $C_i = \int_{-\infty}^{+\infty}\mathrm{d}x'[g(x')A_i(x',0) - f(x')A_{it'}(x',0)]$ or, with (4.9) and (4.1),

$$v_0(x,t) = \int_{-\infty}^{+\infty}\mathrm{d}x'[g(x')G^0(x,t;x',0) - f(x')G^0_{t'}(x,t;x',t')|_{t'=0}]. \tag{4.11}$$

Another useful formulation in the case of non-zero initial values is the following. If we write the total solution v as $v = a + w$, where a is any function that satisfies the initial conditions, and apply the operator L, $Lw = Lv - La = F - La$, then w is given by the Green function solution for vanishing initial values, and we have

$$v(x,t) = a(x,t) + \int_0^t\mathrm{d}t'\int_{-\infty}^{+\infty}\mathrm{d}x' G^0(x,t;x',t')[F(x',t') - L'a(x',t')] \tag{4.12}$$

where L' denotes the operator L in x', t'-coordinates.

The general solution (2.9), $v = v_0 + v_1$, given by (4.11) and (2.7), resembles the result obtained by Riemann's integration method [24, 14]. There are, however, some differences. First, the Riemann function R is defined in a way that $R = 2G^0$. Second, the integration area for x', t' is confined to $|x - x'| \leqslant |t - t'|$, $t' \geqslant 0$. Indeed, it can be shown that $G^0(x,t;x',t') = 0$ for $|x - x'| > |t - t'|$. The integration area in (2.7) and (4.11) may, therefore, be greatly reduced. Because of the discontinuities for $|x - x'| = |t - t'|$, however, when confining oneself to the integration area $|x - x'| \leqslant |t - t'|$, the term $\frac{1}{2}[f(x-t) + f(x+t)]$ should be added to (2.9) (cf also [25]). In [14] it has been shown, independently of [26], that the Riemann function (and therefore, also the Green function) for the perturbed breather problem may be represented in closed form, namely in terms of Lommel functions of two variables. An analogous result holds for the perturbed kink problem, similarly as in [26].

5. Modified perturbation theory. Adiabatic approximation

It has earlier been pointed out [27, 4] that in the perturbed solution v, (2.7), through the integration of the discrete parts of the Green functions (4.7) and (4.8), there may arise 'secular', unphysical terms that grow linearly or even stronger with time t. This is a very general aspect, not confined to soliton theory, and many methods have been developed in order to remedy this unpleasant effect [28]. These methods imply that we have to give up the exact first-order perturbation theory employed so far. We shall adopt the so-called 'two-time-scale' method similarly as in [27, 4]. It seems to us that our procedure of first investigating the pure first-order approximation and then dealing with the modifications is more systematic and evident. Our procedure for the perturbed sine–Gordon equation differs from that of McLaughlin and Scott [4] mainly in that we do not consider a system of two first-order differential equations in place of the familiar second-order differential equation. Thus our treatment appears to be more direct and transparent and, as will be seen, permits further insight.

It is physically obvious to assume that a soliton under a weak perturbation will primarily keep its original form but that the parameters describing it will slowly change with time. So we replace the ansatz (2.2) for the perturbed soliton solution by the new ansatz

$$u = \hat{u}_s + \varepsilon\hat{v} \tag{5.1}$$

where $\hat{u}_s$ denotes the unperturbed soliton form with parameters p_i depending on the 'slow' time $\tau = \varepsilon t$. For products with the 'fast' time t, $P(p_i)t$, appearing in u_s we shall write [27, 4] $\int_0^t P[p_i(\varepsilon t')]\mathrm{d}t'$. The dependences $p_i(\tau)$ will be determined later by proper requirements. The term $\varepsilon\hat{v}$ stands for an additional first-order correction.

Inserting (5.1) into the perturbed sine–Gordon equation (2.1), we have to notice that u_{tt} means total derivatives of $u(t,\tau)$ with respect to t, for instance $u_t = \partial u/\partial t + \varepsilon\partial u/\partial\tau$. Retaining only terms linear in ε, we obtain

$$\hat{v}_{tt} - \hat{v}_{xx} + (\cos\hat{u}_s)\hat{v} = F - F_1 \qquad F_1 = \frac{\partial^2\hat{u}_s}{\partial\tau\,\partial t} + \frac{\partial^2\hat{u}_s}{\partial t\,\partial\tau}. \tag{5.2}$$

The dependence of $\hat{u}_s$ on t and τ leads to a term F_1 which may be considered as an additional force in the linear equation for $\hat{v}$ (the two terms of F_1 are, in general, not the same). If we require $u = u_s$ and $u_t = u_{st}$ for $t = 0$, we have, from (5.1), $\hat{v}(x,0) = 0$, $\hat{v}_t(x,t)|_{t=0} = -\partial\hat{u}_s/\partial\tau|_{t=0}$ as initial conditions for $\hat{v}$. Therefore, the solution of (5.2) is, by means of (2.9), (2.7) and (4.11),

$$\hat{v} = -\int_{-\infty}^{+\infty}\mathrm{d}x'G^0(x,t;x',0)\left.\frac{\partial\hat{u}'_s}{\partial\tau'}\right|_{t'=0} + \int_0^t\mathrm{d}t'\int_{-\infty}^{+\infty}\mathrm{d}x'G^0(x,t;x',t')(F' - F'_1) \tag{5.3}$$

where the primes indicate primed coordinates. We have used here the same Green function as before, which is exact only for constant parameters. Neglecting terms of higher order in ε, however, we may adopt this approximation. Considering now the term $\partial^2\hat{u}'_s/\partial t'\partial\tau'$ of F'_1, we can see that an integration by parts with respect to t' will compensate the first integral of (5.3), and because of $G^0(x,t;x',t) = 0$, (2.8*b*), there remains

$$\hat{v} = \int_0^t\mathrm{d}t'\int_{-\infty}^{+\infty}\mathrm{d}x'\left[G^0(x,t;x',t')\left(F' - \frac{\partial^2\hat{u}'_s}{\partial\tau'\partial t'}\right) + G^0_{t'}(x,t;x',t')\frac{\partial\hat{u}'_s}{\partial\tau'}\right]. \tag{5.4}$$

As seen from (4.7) and (4.8), the Green function G^0 consists of a discrete part and a continuous part, $G^0 = G^0_{\mathrm{d}} + G^0_{\mathrm{c}}$. In order to avoid possible secular terms arising from the discrete part, we now demand that the discrete part of (5.4) be zero:

$$\hat{v}_{\mathrm{d}} \equiv \int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x' \left[G^0_{\mathrm{d}} F' - G^0_{\mathrm{d}} \frac{\partial^2 \hat{u}'_s}{\partial\tau' \partial t'} + G^0_{\mathrm{d}t'} \frac{\partial \hat{u}'_s}{\partial \tau'} \right] = 0. \tag{5.5}$$

This condition determines the functions $p_i(\tau)$. Since G^0_{d} is of the form as shown in (4.8) and the condition (5.5) should hold for all x and t, we obtain conditions valid for each discrete independent function $\varphi_\mu(x, t)$

$$\int_{-\infty}^{+\infty} \mathrm{d}x \left[\varphi_\mu F - \varphi_\mu \frac{\partial^2 \hat{u}_s}{\partial\tau \partial t} + \varphi_{\mu t} \frac{\partial \hat{u}_s}{\partial \tau} \right] = 0. \tag{5.6}$$

Because of the above-mentioned approximate character of G^0, we may allow here the parameters in φ_μ also to modulate with τ, to be denoted as $\hat{\varphi}_\mu$.

If the parameters p_i depend not only on τ but also explicitly on t (which happens, e.g., in the breather case), i.e. both $\mathrm{d}p_i/\mathrm{d}t$ and $\mathrm{d}^2 p_i/\mathrm{d}t^2$ are of order ε, the last term of F_1 in (5.2) is to be understood as the total derivative of $\partial\hat{u}_s/\partial\tau$, and the integration by parts following (5.3) leads to the same results. Instead of $\mathrm{d}p_i/\mathrm{d}\tau$ we then have to write $\varepsilon^{-1}\mathrm{d}p_i/\mathrm{d}t$.

To be specific, we consider the two soliton cases separately.

(1) *Kink case.* As an alternative form of (3.4) we have

$$\hat{u}_k = 4 \arctan \exp \hat{x} \qquad \hat{x} = \left[x - \int_0^t C(\tau')\mathrm{d}t' - x_0(\tau) \right] (1 - C^2(\tau))^{-1/2} \tag{5.7}$$

with two parameters $C(\tau)$ and $x_0(\tau)$. As discrete basis functions we had found $\varphi_1 = \operatorname{sech} \bar{x}$ and $\varphi_2 = \bar{t} \operatorname{sech} \bar{x}$ ($\bar{x}$ results from $\hat{x}$ for $\varepsilon = 0$). The derivatives of $\hat{u}_k$ may be expressed in terms of $\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_{1t}$ and $\hat{\varphi}_{2t}$, and the sums of the second and third term of (5.6) become proportional to the integral $\hat{M}_{21} = \int_{-\infty}^{+\infty} (\hat{\varphi}_1 \hat{\varphi}_{2t} - \hat{\varphi}_{1t} \hat{\varphi}_2)\, \mathrm{d}x = M_{21}$, whose value has already been given ahead of (4.7). The following relations result

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = -\frac{1 - C^2}{4} \int_{-\infty}^{+\infty} \mathrm{d}x \operatorname{sech} \hat{x} F(\hat{x}) \qquad \frac{\mathrm{d}x_0}{\mathrm{d}\tau} = -\frac{C\sqrt{1 - C^2}}{4} \int_{-\infty}^{+\infty} \mathrm{d}x\, \hat{x} \operatorname{sech} \hat{x} F(\hat{x}). \tag{5.8}$$

These relations, also to be written with $\mathrm{d}x = \sqrt{1 - C^2}\, \mathrm{d}\hat{x}$, represent ordinary first-order differential equations for the functions $C(\tau)$ and $x_0(\tau)$. In order that C and x_0 depend, as presupposed, on τ only, we had to assume that F depends on x and t only through $\hat{x}$. In case F depends on $\hat{x}$ and t, C and x_0 would become functions of τ and t. The relations (5.8) are in accordance with the results of McLaughlin and Scott [4]; Karpman and Solov'ev [6] report the same first relation but another result for $\mathrm{d}x_0/\mathrm{d}\tau$.

(2) *Breather case.* We consider the general breather given by (3.11) with (3.12). For typographical convenience, we shall write x, t instead of $\tilde{x}, \tilde{t}$. We allow again the parameters c_1, c_2, c and σ to modulate with $\tau = \varepsilon t$ and assume the modulated breather in the form

$$\hat{u}_b(x, t, \tau) = -4 \arctan(\gamma \sin\theta \operatorname{sech} z) \tag{5.9a}$$

$$z = \alpha r (x - x_1) \qquad x_1 = \int_0^t c(t')\, \mathrm{d}t' + x_0 = T_1 + x_0 \tag{5.9b}$$

$$\theta = \theta_1 - c\gamma^{-1} z \qquad \theta_1 = \int_0^t \beta(t') r^{-1}(t')\, \mathrm{d}t' + \theta_0 = T_2 + \theta_0 \tag{5.9c}$$

where $r = (1 - c^2)^{-1/2}$. The parameters are now $p_i = (x_0, \theta_0, c, \sigma)$. The discrete basis functions φ_μ are obtained by differentiating the function u_b, i.e. (5.9a) with $T_1 = ct$, $T_2 = \beta r^{-1} t$, with respect to p_i. In order that we may utilize the relationships for the elements M_{ij} of the last section, we have to normalize the functions φ_μ such that they coincide with the basis functions in (3.16) when the transformed coordinates (3.12) are introduced. The modified functions $\hat{\varphi}_\mu$ are then given by writing T_1 and T_2 as in (5.9b, c). The derivatives $\partial \hat{u}_b/\partial \tau = \sum_i (\partial \hat{u}_b/\partial p_i) \mathrm{d}p_i/\mathrm{d}\tau$ in (5.6) are connected with the $\hat{\varphi}_\mu$ in the following way:

$$\partial \hat{u}_b/\partial x_0 = -4\beta r(\hat{\varphi}_1 + c\gamma^{-1}\hat{\varphi}_2) \qquad \partial \hat{u}_b/\partial c = -4r(r\hat{\varphi}_3 - \beta c^{-1}\hat{\varphi}_1 T_1)$$
$$\partial \hat{u}_b/\partial \theta_0 = -4\gamma^{-1}\hat{\varphi}_2 \qquad \partial \hat{u}_b/\partial \sigma = 4\alpha(\hat{\varphi}_4 + \hat{\varphi}_2 T_2). \tag{5.10}$$

For the differences $\partial^2 \hat{u}_b/\partial p_i \partial t - \partial^2 \hat{u}_b/\partial t \partial p_i$ one deduces $(0, 0, -\beta c r \partial \hat{u}_b/\partial \theta_0 + \partial \hat{u}_b/\partial x_0, \alpha^2 r^{-1} \partial \hat{u}_b/\partial \theta_0)$ for $p_i = (x_0, \theta_0, c, \sigma)$, respectively. The system (5.6) then assumes the form

$$\int_{-\infty}^{+\infty} \hat{\varphi}_\mu F \, \mathrm{d}x = 4 \sum_{\nu=1}^{4} a_\nu \int_{-\infty}^{+\infty} (\hat{\varphi}_{\nu t}\hat{\varphi}_\mu - \hat{\varphi}_\nu \hat{\varphi}_{\mu t}) \, \mathrm{d}x \tag{5.11}$$

with $a_1 = \beta r(-\mathring{x}_0 + c^{-1} T_1 \mathring{c})$, $a_2 = -\gamma^{-1}(\mathring{\theta}_0 + \beta c r \mathring{x}_0 - \alpha\gamma T_2 \mathring{\sigma})$, $a_3 = -r^2 \mathring{c}$ and $a_4 = \alpha \mathring{\sigma}$, where $^\circ$ means $\mathrm{d}/\mathrm{d}\tau$. The matrix elements on the right-hand side of (5.11), $\hat{M}_{\nu\mu}$, are (as shown in the appendix) independent of c and are equal to those evaluated in the last section for $c = 0$. The non-zero elements are $\hat{M}_{31} = -\hat{M}_{13} = \hat{M}_{42} = -\hat{M}_{24} = \gamma$. For $\mu = 1$ and 2, the relations for $\mathring{c}$ and $\mathring{\sigma}$ follow from (5.11) at once. For $\mu = 3$ and 4, (5.11) reduces to the respective relations for $\mathring{x}_0$ and $\mathring{\theta}_0$

$$\int_{-\infty}^{+\infty} (\partial \hat{u}_b/\partial c, \partial \hat{u}_b/\partial \sigma) F \mathrm{d}x = 4^2 \alpha(-r^3 \mathring{x}_0, \; \mathring{\theta}_0 + \beta c r \, \mathring{x}_0). \tag{5.11a}$$

The parameters p_i turn out to depend not only on τ but also on t. Then replacing $\mathrm{d}/\mathrm{d}\tau$ by $\varepsilon^{-1}\mathrm{d}/\mathrm{d}t$, we obtain the following final results,

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{\varepsilon}{4}\beta^{-1} r^{-3} I_1 \qquad \frac{\mathrm{d}x_1}{\mathrm{d}t} = c - \frac{\varepsilon}{4}(\alpha r)^{-2}[\gamma c I_3 + I_4]$$
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\varepsilon}{4}(\alpha\beta r)^{-1} I_2 \qquad \frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \beta r^{-1} + \frac{\varepsilon}{4}(\alpha r)^{-1}[-r^{-2} I_3 + \alpha^{-2}\gamma c I_4 + \beta^{-2} I_5] \tag{5.12}$$

with

$$I_\nu = \int_{-\infty}^{+\infty} \frac{f_\nu}{\boldsymbol{C}^2 + \gamma^2 \boldsymbol{s}^2} F \mathrm{d}z \qquad f_1 = \boldsymbol{S}\boldsymbol{s} \qquad f_2 = \boldsymbol{C}\boldsymbol{c}$$
$$f_3 = z f_1 \qquad f_4 = z f_2 \qquad f_5 = \boldsymbol{C}\boldsymbol{s}$$

where mean $\boldsymbol{S} = \sinh z$, $\boldsymbol{C} = \cosh z$, $\boldsymbol{s} = \sin\theta$, $\boldsymbol{c} = \cos\theta$ and, as before, $\alpha = \operatorname{sech}\sigma$, $\beta = \tanh\sigma$, $\gamma = \operatorname{csch}\sigma$. The functions $c(t)$, $\sigma(t)$, $x_1(t)$ and $\theta_1(t)$ are obtained by integrating the relations (5.12). Our results (5.12), when transformed into the language of Karpman *et al* [7], are in full agreement with their results obtained by inverse scattering methods. McLaughlin and Scott [4] have not considered the general case but only the stationary breather ($c = 0$) with $F(x, t)$ even in x. Neither have they given explicit results in a general form, but if one carries on their implicit formulation, one also arrives at the specialized results for $\mathrm{d}\sigma/\mathrm{d}t$ and $\mathrm{d}\theta_1/\mathrm{d}t$ in (5.12). These special results have first been derived by Kosevich and Kivshar [29] in a way similar to that of [7].

After having determined the time dependence of the parameters p_j in $\hat{u}_s$ for both kinks and breathers by demanding the discrete part of equation (5.4) to be zero, let us consider the remaining part formed with the continuous part of the Green function, G^0_{c}. There arise integrals like those in (5.6), but now the continuous functions φ_k appear instead of φ_μ. This means that elements $M_{\nu k}$ appear which, in the last section, have been shown to vanish. As a result, the perturbed soliton solution (5.1) now takes the form

$$u = \hat{u}_s + \varepsilon \int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x'\, G^0_{\mathrm{c}}(x, t; x', t') F(x', t'). \tag{5.13}$$

Here $\hat{u}_s$ represents the so-called adiabatic approximation, that is the unperturbed soliton form with time-dependent parameters p_j. A first-order correction to this form is given by the integral with the continuous part of the Green function. The fact that here only the perturbation F appears, and not the 'effective force' $F - F_1$, has not been noted by McLaughlin and Scott [4]. The adiabatic approximation $\hat{u}_s$ represents a nonlinear generalization of the former result (2.2) with the discrete part of (2.7). A linear expansion of $\hat{u}_s$ in ε leads, indeed, to u_s plus the integral term formed with the discrete part G^0_{d} of the Green function.

A particular case that deserves our attention is a stationary breather ($c = 0$, and, e.g., $c_1 = c_2 = 0$) under a constant perturbation $F = S$. A static solution of the sine–Gordon equation, originally $u = 0$, is now $u = \arcsin \varepsilon S \approx \varepsilon S$. Since we are interested in a perturbed solution that tends to the static solution at large distances from the centre of the breather, we take as initial conditions $v(x, 0) = f(x) = S$, $v_t(x, 0) = g(x) = 0$. In this case there are, as stated earlier [13], in principle two possibilities. This depends on whether we adopt the representation as shown in (4.11) or in (4.12). The modified solution (5.1) then becomes, in the first case,

$$u = \hat{u}_{b1} - \int_{-\infty}^{+\infty} \mathrm{d}x'\, G^0_{t'}(x, t; x', 0)\varepsilon S + \int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x'\, G^0_{\mathrm{c}}(x, t; x', t')\varepsilon S \tag{5.14}$$

where in $\hat{u}_b = \hat{u}_{b1}$ the parameters are determined through (5.12) with $F = S$. The first integral in (5.14) which represents the term v_0 of (4.11) in the present special case, may be evaluated in closed form, which is most easily done with the help of the aforementioned key equation [13, 14]. The discrete part of G^0 does not contribute and the result is

$$\int_{-\infty}^{+\infty} \mathrm{d}x'\, G^0_{t'}(x, t; x', 0) = \left(1 - \frac{2(\cosh^2 z - \sin^2\theta)}{\cosh^2 z + \gamma^2 \sin^2\theta}\right)\cos t \, - \frac{\beta^{-1}\sin 2\theta}{\cosh^2 z + \gamma^2 \sin^2\theta}\sin t. \tag{5.15}$$

Here an oscillating term arises for $|z| \to \infty$ (which is compensated by the last term in (5.14)). In the second case, with $a = S$ and $La = S\cos u_b$, we obtain

$$u = \hat{u}_{b2} + \varepsilon S + \int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x'\, G^0_{\mathrm{c}}(x, t; x', t')\varepsilon S(1 - \cos u'_b). \tag{5.16}$$

Here the parameters in $\hat{u}_b = \hat{u}_{b2}$ are to be formed from (5.12) with $F = S(1 - \cos u_b)$. As a result, it appears that the second representation is to be preferred and that $\hat{u}_{b2} + \varepsilon S$ is the appropriate adiabatic approximation. This contrasts with the presentation in [7].

6. Application: radiation of energy

A perturbed soliton keeps, in the sense of the adiabatic approximation, most of its properties; nevertheless it gradually changes during the perturbation and loses energy. The radiation of energy is determined through the second part $\varepsilon v_{\mathrm{c}}$ of the perturbed solution (5.13). As an example, the energy radiation of a breather under a constant perturbing force has been calculated within the present approach in [13, 14].

As another example we consider the interaction of a moving kink with a localized perturbation [3]

$$\varepsilon F = \varepsilon\delta(x)\,\sin u_k = \varepsilon\delta(x)(-2)\,\tanh\bar{x}\,\mathrm{sech}\,\bar{x} \tag{6.1}$$

where the kink solution u_k is given in (3.4) and $\bar{x}$ reads in terms of the velocity parameter V as $\bar{x} = (x - Vt - x_0)(1 - V^2)^{-1/2}$. The total energy of the system with the perturbed solution u is expressed through the Hamiltonian

$$H = \int_{-\infty}^{+\infty} [\tfrac{1}{2}(u_t^2 + u_x^2) + (1 - \cos u)(1 - \varepsilon\delta(x))]\,\mathrm{d}x. \tag{6.2}$$

The kink is assumed to start at $t = 0$ from $x = x_0 < 0$, $|x_0| \gg 1$, and to move to the right ($V > 0$) towards the local inhomogeneity at $x = 0$. We consider the situation at large times ($t \gg 1$) after the kink has passed the impurity and all the radiation energy has been emitted. (The adiabatic approximation (5.8) applied to this case yields, to the order ε, no final change of the velocity V, only a phase shift $\Delta = (\varepsilon/2)(1 - V^2)/V^2$, defined as $\Delta = \int_0^t [V(\tau') - V(0)]\mathrm{d}t' + x_0(\tau) - x_0(0)$ for $t \gg 1$.)

Outside of the kink region the perturbed solution is $\varepsilon\upsilon = \varepsilon\upsilon_{\mathrm{c}}$ and, therefore, the total emitted energy is derived from (6.2) to be (to the order ε^2)

$$E = \frac{\varepsilon^2}{2}\int_{-\infty}^{+\infty} \left(\upsilon_t^2 + \upsilon_x^2 + \upsilon^2\right)\mathrm{d}x. \tag{6.3}$$

From (5.13) and (4.7), the solution υ with the perturbation (6.1) becomes

$$\upsilon = \int_{-\infty}^{+\infty}\mathrm{d}k\int_0^t \mathrm{d}t'\left[\frac{\left(\tanh\bar{x} - \mathrm{i}\bar{k}\right)\left(\tanh\bar{x}' + \mathrm{i}\bar{k}\right)}{4\pi\,\mathrm{i}\,\omega\,\bar{\omega}^2}\mathrm{e}^{\mathrm{i}kx + \mathrm{i}\omega(t-t')} + \{-\omega\}\right](-2)\frac{\tanh\bar{x}'}{\cosh\bar{x}'} \tag{6.4}$$

where now $\bar{x}' = (-Vt' - x_0)(1 - V^2)^{-1/2}$, and $\bar{k}$ and $\bar{\omega}$ are defined in (3.9). The t' integration can be performed exactly. Also the x integration over $\upsilon^2 = \upsilon\upsilon^*$ may be carried out, observing that some terms vanish with the first subsequent k integration. Finally, one obtains

$$E = \frac{\varepsilon^2\pi}{8V^6}(1 - V^2)^2\int_{-\infty}^{+\infty}\mathrm{d}k[(\omega + Vk)^2 + (\omega - Vk)^2]\mathrm{sech}^2(\pi\omega\sqrt{1 - V^2}/2V). \tag{6.5}$$

Considering the wave expressions from which the above contributions originate, one may separate the total energy into two parts, corresponding to the energy radiated to the left (←) and to the right (→), respectively,

$$E_{\rightarrow}^{\leftarrow} = \frac{\varepsilon^2\pi}{4V^6}(1 - V^2)^2\int_0^{\infty}\mathrm{d}k(\omega \pm Vk)^2\mathrm{sech}^2(\pi\omega\sqrt{1 - V^2}/2V). \tag{6.6}$$

As can be seen, most of the energy is radiated to the left (backwards), irrespective of the sign of the impurity.

The total emitted energy (6.5) can be evaluated in closed form in two limiting cases:

$$|V| \ll 1:\; E = \varepsilon^2\sqrt{2}\pi|V|^{-11/2}\exp(-\pi/|V|) \tag{6.7a}$$

$$\sqrt{1 - V^2} \ll 1:\; E = \varepsilon^2(2/3)\sqrt{1 - V^2}. \tag{6.7b}$$

These results agree with those of Kivshar and Malomed [3, 30], obtained by inverse scattering methods.

For a more general confined non-dissipative perturbation εF the radiated energy E from a kink may be formulated as

$$E = \int_{-\infty}^{+\infty} \mathrm{d}k \left[\frac{\varepsilon^2 |I(k)|^2}{8\pi \bar{\omega}^2} + \{-\omega\} \right]$$
$$I(k) = \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x (\tanh \bar{x} + \mathrm{i}\bar{k}) \mathrm{e}^{-\mathrm{i}(kx+\omega t)} F(x,t) \tag{6.8}$$

provided the time integral converges. This expression corresponds to that derived from inverse scattering theory [3].

7. Summary and conclusion

The intention of the present paper has been to show that it is possible to establish a systematic perturbation theory for sine–Gordon solitons without having to use or to borrow from inverse scattering theory. Our treatment is based on a Green function formalism with special attention to the initial-value problem. The basis functions for an expansion of the Green function are constructed either directly or by utilizing the Bäcklund transformation. The main problem, the determination of the Green functions, is solved by employing Green's theorem. This is, in this connection, a novel and rather general procedure and works equally well for one-soliton and multi-soliton solutions. The present treatment also allows one to reproduce the adiabatic approximation in a consistent way and to discuss and complement the main papers in this field.

As compared to the papers of Karpman and Solov'ev ([6]), Kosevich and Kivshar ([29]), Karpman, Maslov and Solov'ev ([7]) and McLaughlin and Scott ([4]), the present treatment is complete in that it derives for both kinks and general breathers explicit expressions for the first-order solutions as well as the adiabatic modulations of the parameters. As distinguished from [4], we have considered the familiar second-order sine–Gordon equation and not a system of two first-order differential equations, which obviously permits a clearer representation. While [6], [29], and [7] take full account of inverse scattering theory, [4] use such methods only for determining the Green functions, but their derivation ([4], appendix; cf also Kaup [18]) appears to be much more involved than ours utilizing Green's theorem in a simple way. As opposed to [4], we could show that in the continuous part of the perturbed solution only the true perturbation term enters and not an 'effective force'.

Our results concerning the adiabatic approximation for perturbed kinks agree with those of [4], but one relation differs from [6]. Unlike these investigations, Herman [19] studied perturbed kinks in light-cone coordinates only. For the general breather we have obtained full agreement with [7]; [4] and [29] have considered merely the stationary breather, but only [29] have presented explicit relations. In the case of a constant perturbation we propose another adiabatic approximation than that following from [7].

In an application to a simple example of a kink scattered by a local inhomogeneity we have shown that the present theory is well suited to the calculation of the energy emission from perturbed solitons.

In conclusion, it seems to us that the present approach is of a similar terseness and efficiency as the treatments employing inverse scattering theory but is distinguished by its relative simplicity and transparency.

Acknowledgment

The author gratefully acknowledges the continual and stimulating interest of Professor A Seeger in the present work.

Appendix. Invariance of matrix elements M_{ij}

It is to be shown that the elements M_{ij} defined in section 4 are independent of the Lorentz parameter c. We consider

$$M_{ij}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x\, [\varphi_{it}(\bar{x},\bar{t})\varphi_j^*(\bar{x},\bar{t}) - \varphi_i(\bar{x},\bar{t})\varphi_{jt}^*(\bar{x},\bar{t})] \tag{A1}$$

with $\bar{x} = r(x - ct), \bar{t} = r(t - cx), r = (1 - c^2)^{-1/2}$, and we form $\partial M_{ij}/\partial c$. With $\partial\varphi/\partial c = -r^2(\bar{t}\varphi_{\bar{x}} + \bar{x}\varphi_{\bar{t}}) = -r^2(t\varphi_x + x\varphi_t)$ we get

$$\frac{\partial}{\partial c}[\varphi_{it}\varphi_j^* - \varphi_i\varphi_{jt}^*] = -r^2\left[t\frac{\partial}{\partial x}(\varphi_{it}\varphi_j^* - \varphi_i\varphi_{jt}^*) + \varphi_{ix}\varphi_j^* - \varphi_i\varphi_{jx}^* + x(\varphi_{itt}\varphi_j^* - \varphi_i\varphi_{jtt}^*)\right].$$

An integration by parts gives with (3.1) the result

$$\frac{\partial}{\partial c}M_{ij} = -r^2[t(\varphi_{it}\varphi_j^* - \varphi_i\varphi_{jt}^*) + x(\varphi_{ix}\varphi_j^* - \varphi_i\varphi_{jx}^*)]|_{x=-\infty}^{x=+\infty}. \tag{A2}$$

This expression vanishes for i or j representing a discrete state. If both i and j denote continuous states, $\partial M_{ij}/\partial c$ becomes zero with the subsequent k integration indicated in the first relation of (4.4).

References

[1] Seeger A and Kochendörfer A 1951 *Z. Phys.* **130** 321
[2] Mann E 1994 *Nonlinear Coherent Structures in Physics and Biology (NATO ASI Series B 329)* ed K H Spatschek and F G Mertens (New York: Plenum) p 317
[3] Kivshar Yu S and Malomed B A 1989 *Rev. Mod. Phys.* **61** 763
[4] McLaughlin D W and Scott A C 1978 *Phys. Rev.* A **18** 1652
[5] Kaup D J and Newell A C 1978 *Proc. R. Soc. London* A **361** 413
[6] Karpman V I and Solov'ev V V 1981 *Phys. Lett.* **84A** 39
[7] Karpman V I, Maslov E M and Solov'ev V V 1983 *Sov. Phys.–JETP* **57** 167
[8] Fogel M B, Trullinger S E, Bishop A R and Krumhansl J A 1977 *Phys. Rev.* B **15** 1578
[9] Reinisch G and Fernandez J C 1981 *Phys. Rev.* B **24** 835
[10] Kodama Y and Ablowitz M J 1981 *Stud. Appl. Math.* **64** 225
[11] Seeger A, Donth H and Kochendörfer A 1953 *Z. Phys.* **134** 173
[12] Seeger A 1980 Solitons in crystals *Continuum Models of Discrete Systems* ed E Kröner and K H Anthony (Waterloo, Ontario: University of Waterloo Press) p 253
[13] Mann E 1991 *Nonlinear Coherent Structures in Physics and Biology (Lecture Notes in Physics 393)* ed M Remoissenet and M Peyrard (Berlin: Springer) p 351
[14] Mann E 1996 *Teor. Mat. Fiz.* **107** 439
Mann E 1996 *Theor. Math. Phys.* to appear
[15] Mann E 1997 *J. Math. Phys.* to appear
[16] Newell A C 1980 *Solitons* ed R K Bullough and P J Caudrey (Berlin: Springer) p 177
[17] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)
[18] Kaup D J 1984 *J. Math. Phys.* **25** 2467
[19] Herman R L 1990 *J. Phys. A: Math. Gen.* **23** 2327
[20] Döttling R, Esslinger J, Lay W and Seeger A 1990 *Nonlinear Coherent Structures (Lecture Notes in Physics 353)* ed M Barthes and J Léon (Berlin: Springer) p 193
[21] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* part I (New York: McGraw-Hill)
[22] Mann E 1987 *Phys. Status Solidi* b **144** 115

[23] Lamb G L Jr 1980 *Elements of Soliton Theory* (New York: Wiley)
[24] Courant R and Hilbert D 1968 *Methoden der Mathematischen Physik* vol II (Berlin: Springer)
[25] Mackie AG 1989 *Boundary Value Problems* (Edinburgh: Scottish Academic Press)
[26] Flesch R J and Trullinger S E 1987 *J. Math. Phys.* **28** 1619
[27] Keener J P and McLaughlin D W 1977 *Phys. Rev.* A **16** 777
[28] Kevorkian J and Cole J D 1981 *Perturbation Methods in Applied Mathematics* (New York: Springer)
[29] Kosevich A M and Kivshar Yu S 1982 *Sov. J. Low Temp. Phys.* **8** 644
[30] Kivshar Yu S and Malomed B A 1985 *Phys. Lett.* **111A** 427